

Vector Worksheet

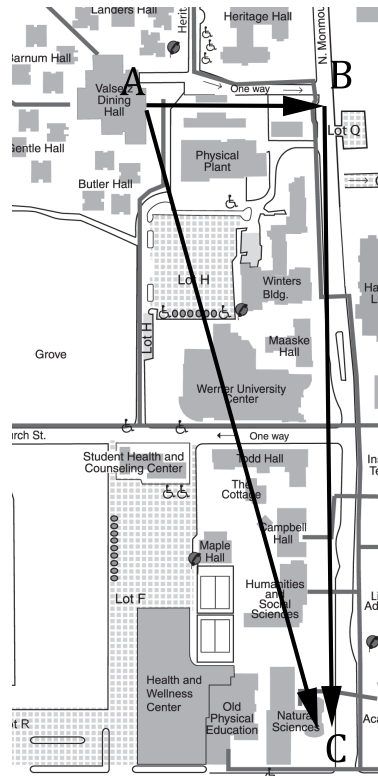
Much of the physical world can be described in terms of numbers. Examples of this are the mass of an object, its temperature and its volume. These are called scalar quantities. But some quantities also need a direction to fully describe it. What if you wanted directions to the nearest physics lab? The answer "Go three miles" is not very helpful. More useful would be "Go three miles south." A quantity having both size (magnitude) and direction is called a vector quantity. Vectors are not just about position. Such quantities as velocity, acceleration, force and momentum all have both a size and a direction. For example, what we know as "speed" is just the magnitude of the velocity vector. To fully describe velocity, we must have both the speed and the direction in which we are going.

We represent vectors geometrically with an arrow. This arrow will have a length that represents magnitude and point in a direction. This worksheet will walk you through some basic vector operations. In your textbooks, you will see vectors denoted in boldface (\mathbf{v}), but when writing a vector, we denote it by writing an arrow above the letter (\vec{v}). We can also, and will do here, use this notation: \vec{v}_{AB} to denote a vector that has its tail at a point A and the tip of its arrow at another point B . In this case we say "The vector from A to B ."

Vector Facts

There are many different types of vector quantities, but they all have similar properties. To explore these properties, let's look at one type of vector quantity, position vectors.

Suppose you are trying to get from Valsetz Dining Hall to the Natural Sciences Building. On the map below, you begin at Valsetz (point A) and you walk to Monmouth Ave., due east, about 400 feet (point B). The vector \vec{v}_{AB} describes this displacement by starting where you started and ending at the place you will turn. It points in the direction you will go, and has a length (magnitude) that represents the 400 feet you will walk. When you reach point B , you turn and walk down Monmouth Avenue about 1500 feet, and end up at point C , in front of the Natural Sciences Building where you will have your favorite class, physics. Vector \vec{v}_{BC} represents this displacement in the same way as \vec{v}_{AB} , but with the new direction and magnitude. If you were a bird and you wanted to fly from Valsetz to NSS, you would probably fly along vector \vec{v}_{AC} , which describes your flight path, stating point, ending point, direction, and magnitude in much the same way.



As you can see, vectors represent some very important information and, once you get used to using them, will be instrumental in helping you to visualize and solve most physics problems.

Vector Addition and Subtraction

Back to the map above. You start out at point, A , and move to point B . Then you turn and go to point C . If you were the bird, you could have accomplished the same net direction by flying directly from A to C . The resulting displacement vector \vec{v}_{AC} is the sum of \vec{v}_{AB} and \vec{v}_{BC} and is written $\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$. This is illustrated in Figure 2.

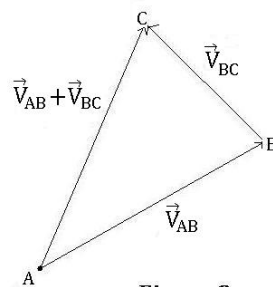
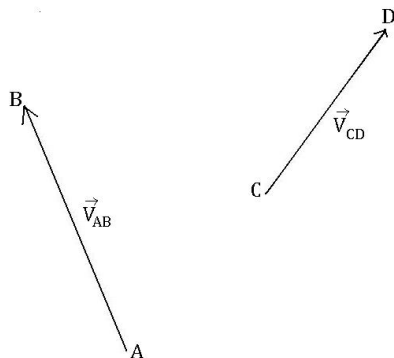


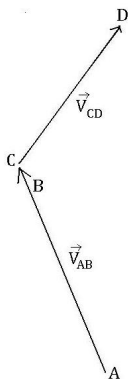
Figure 2

Figure two is useful when we are presented with vectors that are placed with the tip of one

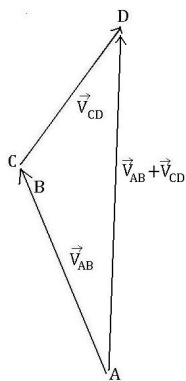
vector at the tail of another. This placement is common when we are discussing position, but more often we want to add vectors that look more like this:



What do we do then? Well, we can move vectors, as long as we don't change their magnitude or direction so that the tip of one is at the tail of the other, like this:



Now we can draw the vector that is $\vec{V}_{AB} + \vec{V}_{CD}$:



We can also subtract vectors. This is not very different from vector addition. Really, if we have two vectors, \vec{v}_A and \vec{v}_B , and we want to know $\vec{v}_A - \vec{v}_B$, all we do is add to \vec{v}_A a vector that is in the direction exactly opposite of \vec{v}_B . Thus, $\vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$. Figure 3 illustrates this point.

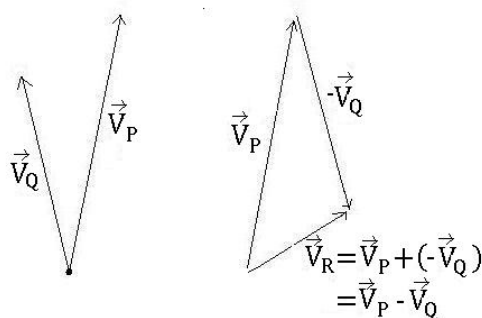


Figure 3

In subtracting vectors, like adding them, if we are presented with vectors that are not placed "tip to tail", we can move them so they are, and then perform our calculations.

Vector Components

Drawing pictures of vectors is great to get a sense of what we have been learning conceptually, but to obtain accurate results, we have to use some math, specifically trigonometry and geometry. Going back to our map above, we can place a coordinate system over the map, with the origin at Valsetz, and see that \vec{v}_{AB} represents motion along the x -axis, and \vec{v}_{BC} represents displacement along the y -axis. The resultant vector \vec{v}_C has *components* in both the x and y directions. Thus, if \vec{v}_x and \vec{v}_y are the x and y components of \vec{v} , then $\vec{v} = \vec{v}_x + \vec{v}_y$. \vec{v}_x and \vec{v}_y are called *component vectors*. The *x-component vector* is the projection of \vec{v} along the x -axis, and the *y-component vector* is the projection of \vec{v} along the y -axis. To visualize a projection, imagine a flashlight on the vector pointing from top to bottom will leave a shadow, or projection, on the x -axis. Figure 4 will be of use to shed some light on this idea.

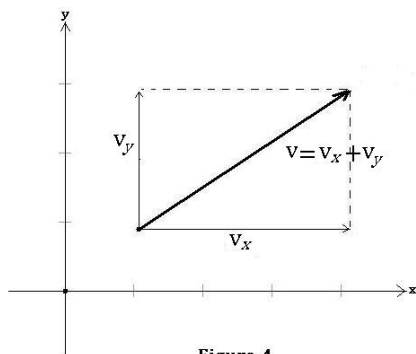
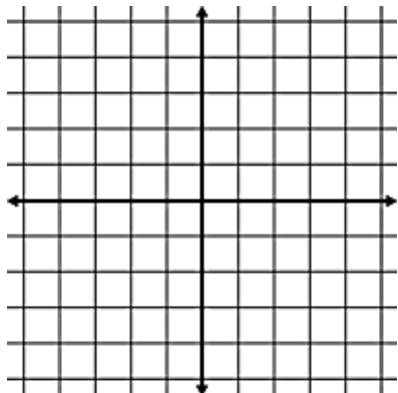


Figure 4

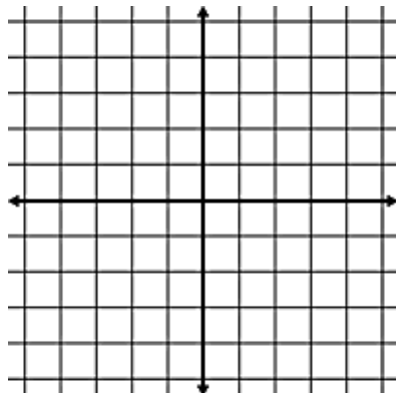
Here's how you determine the components of a vector, \vec{v} :

1. The absolute value $|v_x|$ of the x -component of \vec{v} is the magnitude of \vec{v}_x .
2. The *sign* of $|v_x|$ is positive if \vec{v}_x points in the positive x direction, and negative if \vec{v}_x points in the negative x direction.
3. The y -component \vec{v}_y is determined in the same way.

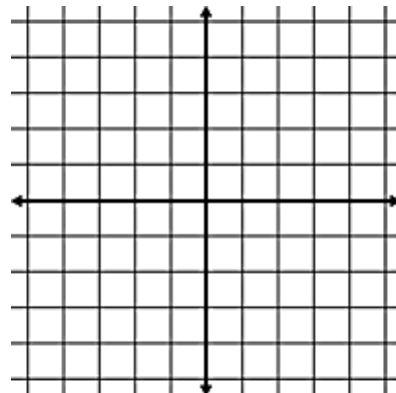
Let's practice drawing some vectors. On the axes below, start at the origin and draw the given component vectors and the resultant vector. Each line is one unit.



(a) $|v_x| = 2, |v_y| = 3$



(b) $|v_x| = -4, |v_y| = 4$



(c) $|v_x| = -3, |v_y| = -5$

Now, let's talk about how one finds the exact numerical values for the magnitudes of vectors. Consider the vector in figure 5.

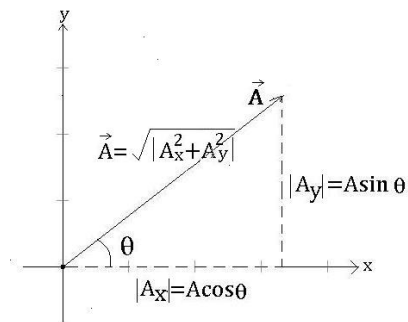


Figure 5

Let's place our \vec{A}_x component vector (see figure 5) along the x -axis. Let's also place our \vec{A}_y component vector so that it runs from the x -axis to the tip of \vec{A} . Why did we do this? We did this because this placement of the component vectors makes the computations for finding the length of them (and the length of \vec{A}) much easier. Notice also that \vec{A} has an angle of θ above the x -axis. Trigonometry gives us the lengths of our component vectors, and since our component vectors form a right triangle with \vec{A} as its hypotenuse, the Pythagorean Theorem can be used to find the length of \vec{A} . An example may help.

Example:

Find the length of the x and y components of vector \vec{R} if \vec{R} is at an angle of 30° above the x -axis and has a magnitude of 25 cm.

Solution:

$$|R_x| = R \cos \theta = 25 \cos(30) = 21.65 \text{ cm}$$

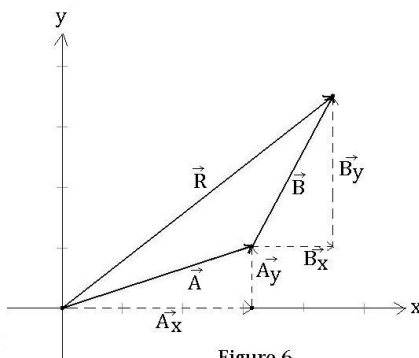
$$|R_y| = R \sin \theta = 25 \sin(30) = 12.5 \text{ cm}$$

Is this solution correct? We can check it using the Pythagorean Theorem:

$$\vec{R} = \sqrt{|R_x|^2 + |R_y|^2}, \text{ so } \sqrt{21.65^2 + 12.5^2} = 25 \text{ cm. Check!}$$

Going the other way, we can also determine the length and angle of a vector from its x and y components. As in the check of the example above, we can use Pythagorean Theorem to find the length of a vector. To find the angle, we can use the fact that the tangent of angle θ is the ratio of the opposite side to the adjacent side (SOHCAHTOA, remember?), so $\theta = \tan^{-1} \left(\frac{|R_y|}{|R_x|} \right)$.

Using components, we can now add (and subtract) vectors exactly. Thus, if $\vec{R} = \vec{A} + \vec{B}$, then $|R_x| = |A_x + B_x|$ and $|R_y| = |A_y + B_y|$. See figure 6.



Here's an example:

Example:

Find $\vec{R} = \vec{A} + \vec{B}$, given that \vec{A} has magnitude 4 and is at an angle of 30° above the positive x -axis and \vec{B} has magnitude 2 and is at an angle of 55° above the positive x -axis.

Solution:

First find the x and y components of \vec{A} and \vec{B} .

$$|A_x| = A \cos \theta_A = 4 \cos(30) = 3.46$$

$$|A_y| = A \sin \theta_A = 4 \sin(30) = 2$$

$$|B_x| = B \cos \theta_B = 2 \cos(55) = 1.15$$

$$|B_y| = B \sin \theta_B = 2 \sin(55) = 1.64$$

Now add them to find the x and y components of \vec{R} .

$$|R_x| = |A_x + B_x| = 4.61 \qquad |R_y| = |A_y + B_y| = 3.64$$

Use the Pythagorean Theorem to find the length of \vec{R} .

$$|R| = \sqrt{|R_x^2 + R_y^2|} = \sqrt{(4.61)^2 + (3.64)^2} = 5.87$$

Now use \tan to find the measure of \vec{R} above the x -axis.

$$\tan \theta = \frac{|R_y|}{|R_x|} = \frac{3.64}{4.61} = 0.790.$$

$$\tan^{-1}(0.790) = 38.3^\circ.$$

Thus, \vec{R} has a magnitude 5.87 and is at an angle of 38.3° above the positive x -axis.

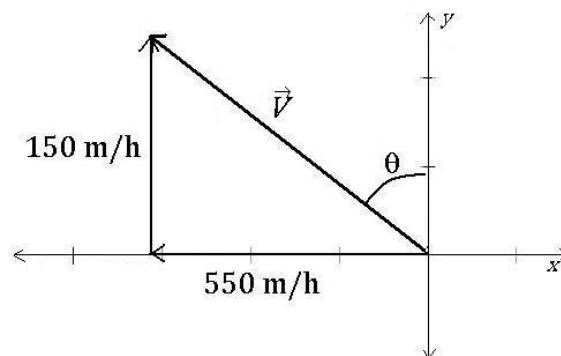
So far, we have only been measuring our angle θ from the positive x -axis. However, using the laws of Trigonometry, we can easily measure θ from any axis we choose. This is simply a matter of keeping track of the component vectors and where they are in relation to the angle of measurement.

Lets say, for example, that you are on a jet plane that takes off from PDX. Its velocity is 550 miles per hour due east. There is a wind blowing with a velocity of 150 miles per hour from the south.

A) Use vector addition to diagram the two vectors and calculate the resultant vector.

B) What is the direction of the jet's velocity vector measured east of north?

The first step in solving any physics problem is to draw a diagram including all of the relevant information. So lets do that:



This diagram gives us a clear picture of the information we have been given in the problem and the information we are looking for.

To solve part (A), we use what we know about adding vector components.

$$|V| = \sqrt{|V_x^2 + |V_y^2|} = \sqrt{(-550)^2 + 150^2} = 570m/h$$

To solve part (B), we need to be careful. Notice that our angle is being measured from the

y -axis toward the negative x -axis. This means that our x component is the side opposite θ and the y component is adjacent to θ . Thus, our calculation for the angle measured east of north looks like this:

$$\tan \theta = \left(\frac{V_x}{V_y} \right)$$

Thus,

$$\theta = \tan^{-1} \left(\frac{V_x}{V_y} \right) = 74.74^\circ$$

So we can conclude that the jet is moving at 570 m/h at an angle of 74.74° east of north.

Let's tie all of these ideas up in a nice, fun physics problem.

You may be familiar with Newton's Second Law of Motion, $\Sigma \vec{F} = m\vec{a}$. In English, this equation says: The sum, or net, (Σ) of the forces (\vec{F}) acting upon an object equals (=) the mass (m) of the object multiplied by the object's acceleration (\vec{a}). Even if you are familiar with this famous equation, did you notice the arrows above F and a before? If so, you may have wondered what they mean. They are vectors! This means that they have both a magnitude and a direction and can be expressed in terms of their components. These arrows are a flashing red light that alerts you to the fact that these quantities imply *two* equations; one in the x direction and one in the y direction.

We'll work through a problem together:

A pack of four Arctic wolves are exerting four different forces upon the carcass of a 500 kg dead polar bear. Wolf one is pulling on the bear with a force of 20 Newtons (N) due north. Wolf two is pulling on the bear with a force of 30 N, 30 degrees north of west. Wolf three is pulling on the bear with a force of 25 N due west, and wolf four is pulling on the bear with a force of 35 N, 10 degrees west of south. Determine the net force acting upon the polar bear and the direction of that force measured north of west. Then compute the magnitude of the acceleration of the polar bear.

Wow. This much information can be scary at first but we are just going to use the techniques we have learned above and work our way through. Take a deep breath. You will be fine.

Your first step should always be to draw a diagram of the situation. Do so here: (Hint: Draw this as though you are looking down on it from directly above with the bear at the origin of your coordinate system, like a two-dimensional tug-o'-war.

We are trying to find the length (magnitude) of the total force vector and its direction. To find the length, we will need to find the x and y components of each of the forces being exerted on the bear. Lets take it a vector at a time.

Wolf One

$$|F_x|=$$

$$|F_y|=$$

Wolf Two

$$|F_x|=$$

$$|F_y|=$$

Wolf Three

$$|F_x|=$$

$$|F_y|=$$

Wolf Four

$$|F_x|=$$

$$|F_y|=$$

Now add up all of the x components: (Careful of minus signs)

Add up all of the y components: (Careful of minus signs)

Now use Pythagorean Theorem to find the magnitude of the total force vector:

What direction is the total force in?

Now find the magnitude of the acceleration by using the equation $\Sigma|F| = m|a|$ and what you know about solving for stuff.

A word on notation:

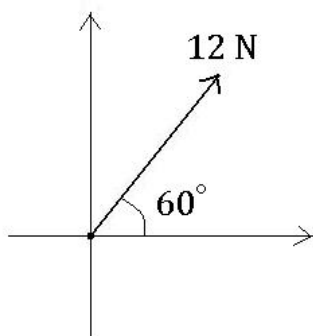
So far we have been expressing vectors and their components as $\vec{v} = \vec{v}_x + \vec{v}_y$. This notation is acceptable but you may run into something that looks like this: $\vec{v} = \langle v_x, v_y \rangle$. This notation is more formal, but accomplishes the same thing. In this new notation, the wolf problem above would look like this:

$$\begin{aligned}\Sigma\vec{F} &= \langle \Sigma F_x, \Sigma F_y \rangle \\ &= \langle F_{1x} + F_{2x} + F_{3x} + F_{4x}, F_{1y} + F_{2y} + F_{3y} + F_{4y} \rangle\end{aligned}$$

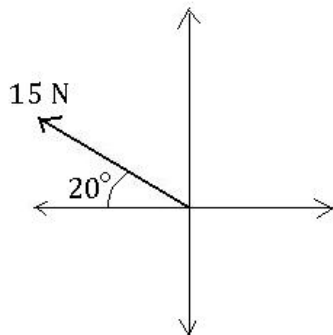
In place of all of the F_{ix} and F_{iy} 's, you would put the numbers you found in the problem and calculate the same way.

Now that you have the idea, here are some practice problems to cement the concepts in your mind.

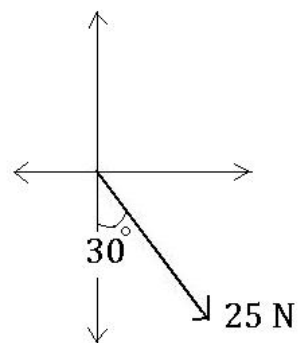
Practice Problems



(a)



(b)



(c)

1. Determine the x and y components of each of the force vectors above.

2. Are the following quantities vectors or scalars? Explain.

- (a) The cost of a theater ticket.
- (b) The current in a river.
- (c) The initial flight path from Houston to Dallas.
- (d) The population of the world.

3. Of the following quantities, which are vectors? If it is, explain why. If it isn't, explain why.

- (a) The speed of a car.
- (b) The velocity of a car.
- (c) How long it takes to do a physics assignment.
- (d) The magnitude of the force required to push a pumpkin off an abandoned overpass.
- (e) The acceleration of said pumpkin.

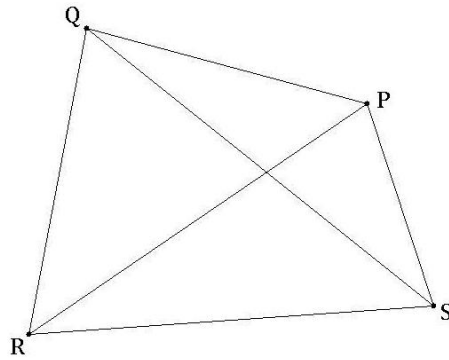
4. Write each combination of vectors as a single vector. See figure below.

(a) $\overrightarrow{PQ} + \overrightarrow{QR}$

(b) $\overrightarrow{RP} + \overrightarrow{PS}$

(c) $\overrightarrow{QS} - \overrightarrow{PS}$

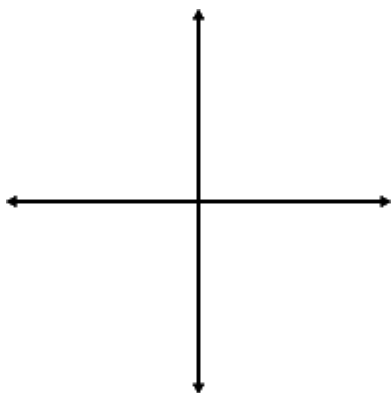
(d) $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ}$



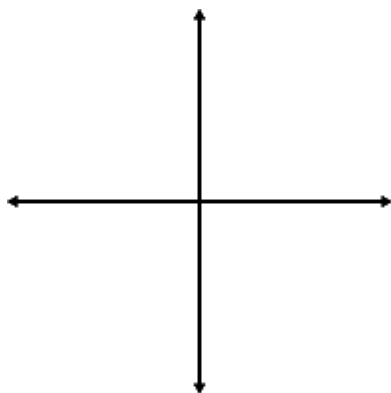
5. If a child pulls a sled through the snow with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

6. A plane flies with a velocity of 52 m/s east through a 12 m/s cross wind blowing the plane south. Find the magnitude and direction (relative to due north) of the resultant velocity at which it travels.

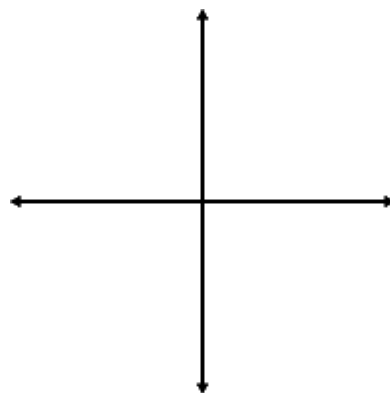
7. Draw the vector \vec{v} and find the magnitude and direction, counter clockwise from the $+x$ -axis.



(a) $v_x=2.5, v_y=1$

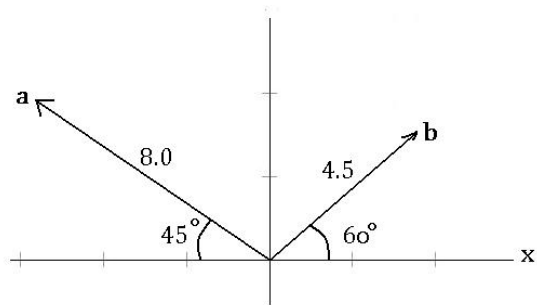


(b) $v_x=-0.5, v_y=1$

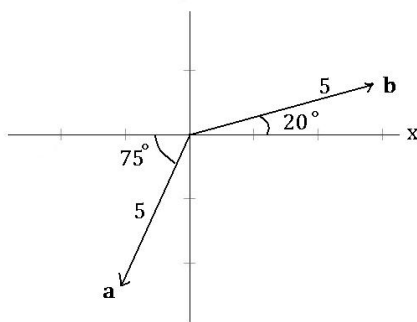


(c) $v_x=-1.5, v_y=-1$

8. Construct $\mathbf{c}=\mathbf{a}+\mathbf{b}$ by drawing and calculating the direction and magnitude of \mathbf{c} . The direction should be measured from the $+x$ -axis.

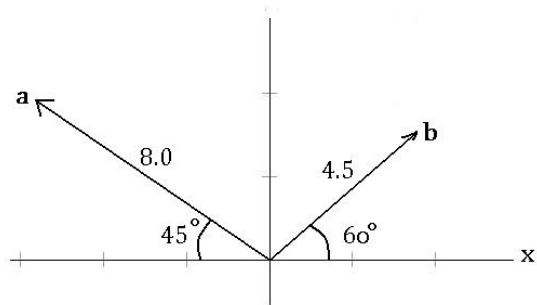


(a)

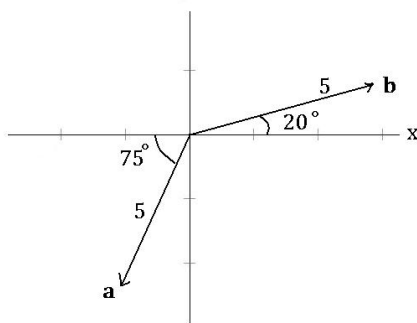


(b)

9. Construct $\mathbf{c}=\mathbf{a}-\mathbf{b}$ by drawing and calculating the direction and magnitude of \mathbf{c} . The direction should be measured from the $+x$ -axis.



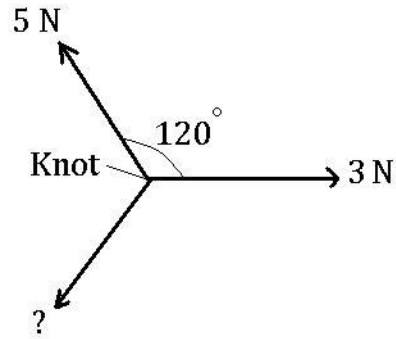
(a)



(b)

10. A racing car is accelerating at 20 m/s^2 70° N of W. Find the acceleration of the car in the north direction and in the west direction.

11. Your uncle Mike's boat can travel 4.0 m/s in still water. One sunny afternoon, you and Unk Mike decide to go fishing. While waiting for a bite, you begin thinking, "If this river is flowing at 5.5 m/s southward, and we are heading eastward, directly across the river, what are the direction and magnitude of our total velocity?" Answer your own question.



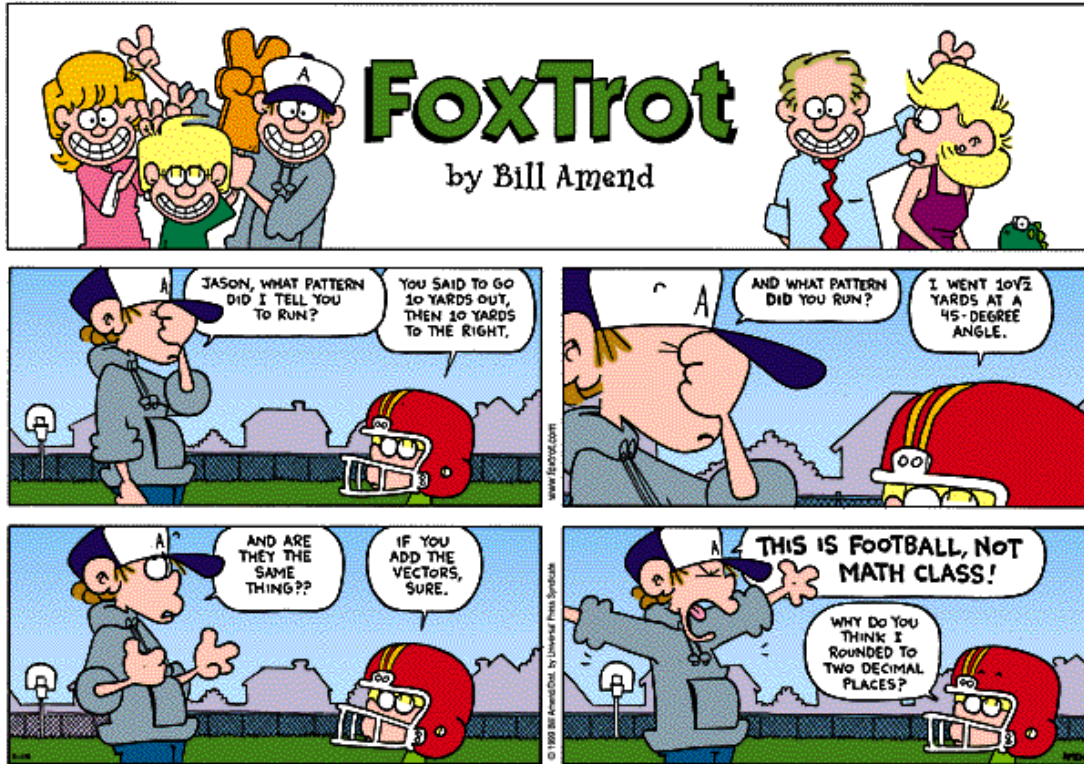
12. The figure above shows three ropes tied together in a knot. One of your friends pulls on a rope with a force of 3 Newtons and another pulls on a second rope with a force of 5 Newtons. How hard and in what direction must you pull on the third rope to keep the knot from moving? (*Hint:* If you don't want the knot to move, the net force must be zero.)

References:

Knight, Randall D., *Physics: A Strategic Approach*, Volume one, Pearson Education, 2004.

Stewart, James, *Calculus: Concepts and Contexts*, Third Edition, Thompson Brooks/Cole, 2005.

The Physics Hypertextbook, Vector Addition and Subtraction, <http://physics.info/vector-addition/>.



Worksheet created by Laura K. Waight